Tentamen Functionaalanalyse
30/08/04

1. Let $F : L^2[0,1] \to \mathbb{C}$ be defined by

$$F(f) := \int_0^1 tf(t)dt, \qquad f \in L^2[0,1].$$

   (a) Is F linear? Justify the answer!

   (b) Show that F is bounded. Determine $\|F\|$.

   (c) Let $G : L^2[0,1] \to \mathbb{C}$ be a continuous linear functional defined on $L^2[0,1]$. Does there exist some $g \in L^2[0,1]$ such that G is of the form

$$G(f) = 3\int_0^1 f(t)g(t)dt, \quad f \in L^2[0,1]?$$

   Justify the answer!

2. Solve the integral equation

$$x(t) - \int_0^\pi x(s)\sin(t+s)ds = 7\sin t, \quad x \in C[0,\pi].$$

3. Let $\mathfrak{H}$ be a Hilbert space and let T and S be linear operators on $\mathfrak{H}$ for which

$$(Tf, g) = (f, Sg), \quad f, g \in \mathfrak{H}.$$

   Show that T, S are bounded operators, and that $S = T^*$.
   Hint. Use the closed graph theorem.

4. Provide the linear space $C^1[0,1]$ with

$$\|x\|_a := \|x\|_\infty + \|x'\|_\infty + |x(0)|, \quad x \in C^1[0,1].$$

   (a) Show that $\|\cdot\|_a$ is a norm on $C^1[0,1]$.

   (b) Show that $C^1[0,1]$ with the norm $\|\cdot\|_a$ is a Banach space.

   (c) Let

$$\|x\|_1 := \max\{\|x\|_\infty, \|x'\|_\infty\}, \quad x \in C^1[0,1].$$

   Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_a$ are equivalent on $C^1[0,1]$.